FINANCE WEB APP
TERM STRUCTURE FITTING

Model fitting to market term structure

THE aim of this web app is to fit the term structure of spot rates to market data. The outputs are optimal parameters for the interpolation of a term structure data-set by a given interpolation method. The user can choose between 4 different parametric models: Vasicek model (Vasicek (1977)), CIR model (Cox, Ingersoll Jr, and Ross (2005)), Nelson & Siegel model (Nelson and Siegel (1987)) or Svensson model (Svensson (1994)). The user can download the fitted spot and zero-coupon bond term structures.

Background information

Calibration framework

Given a market quantity $\mathbf{Y}^{MKT}$ and a parametric model $\boldsymbol{m}(\boldsymbol{\theta})$, model calibration consists in choosing the parameter set $\boldsymbol{\theta}$ that minimizes the distance between a cross section of model quantities (in this case yields and prices) and corresponding market observables

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \left\| \mathbf{Y}^{MOD}(\boldsymbol{\theta}) - \mathbf{Y}^{MKT} \right\| \tag{1}$$

where MOD refers to a model quantity and MKT to a quantity observed on the market. In general, the model parameters are estimated by using a quadratic loss function, i.e., minimizing the difference between market and model quantities in a least-square sense. In particular, different loss functions can be adopted, depending on the use of absolute or percentage errors, that can be computed using prices or yields.

The web app aims to perform the above calibration exercise with reference to market observation of spot rates or zero-coupon bond prices. We let $P_{mkt}(t,T)$ be the market price at time t of a zero-coupon bond expiring in T. Given the zcb price, we can compute the corresponding (continuously compounded) spot rate as $R_{mkt}(t,T) = -\frac{\ln(P_{mkt}(t,T))}{T-t}$. The standard procedure is then to postulate a parametric functional form for the discount curve and then to estimate the parameters of this functional form by solving the problem in 1.

To this aim we define the corresponding model prices $P_{mod}(t,T;\boldsymbol{\theta})$ and rates $R_{mod}(t,T;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ refers to the parameter set of the given model.

Given the above quantities, the model calibration consists in solving the following minimization problem

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(R_{mod}(t,T_i;\boldsymbol{\theta}) - R_{mkt}(t,T_i)\right)^2. \tag{2}$$

The fitting can also be defined in terms of prices

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(P_{mod}(t,T_i;\boldsymbol{\theta}) - P_{mkt}(t,T_i)\right)^2. \tag{3}$$

Minimising price errors sometimes results in fairly large yield errors for short maturities. This is because spot rates are very sensitive to prices for short maturities.

The web app allows the user to perform the calibration using the following models: Nelson & Siegel (NS), Svenson (Sv), Cox, Ingersoll and Ross (CIR) and Vasicek (V). We also allow for a perfect fit of the term structure by using two non-parametric interpolations, i.e. piecewise linear spot rate curve and piecewise linear forward rate curve. We shortly describe these models.

The Nelson & Siegel model

The Nelson & Siegel [3] is a popular model for the discount curve

$$P(t,t+\tau;\boldsymbol{\theta}) = \exp\left(-\tau \times R(t,t+\tau;\boldsymbol{\theta})\right)$$

where the continuously compounded spot rate $R(t,t+\tau)$ has a parsimonious specification

$$R_{NS}(t,t+\tau;\boldsymbol{\theta}) = \beta_0 + \beta_1\left(\frac{1-\exp(-\frac{\tau}{\kappa})}{\frac{\tau}{\kappa}}\right) + \beta_2\left(\frac{1-\exp(-\frac{\tau}{\kappa})}{\frac{\tau}{\kappa}} - \exp(-\frac{\tau}{\kappa})\right) \tag{4}$$

and $\boldsymbol{\theta} = \{\beta_0, \beta_1, \beta_2, \kappa\}$ is the vector of unknown parameters to be estimated. The short and long end of the curve are related to the NS parameters. Indeed

- β_0 specifies the long rate to which the spot rate converges asymptotically

$$\lim_{\tau\to\infty} R_{NS}(t,t+\tau;\boldsymbol{\theta}) = \beta_0.$$

According to formula 4, a change of this parameter will affect all spot rates in the same way. Therefore, β_0 is called level parameter.

- β_1 determines the short term rate, indeed we have
$$\lim_{\tau\to 0} R_{NS}(t, t+\tau; \boldsymbol{\theta}) = \beta_0 + \beta_1.$$
According to formula 4, a change of β_1 affects spot rates according to the loading factor $\left(\frac{1-\exp(-\frac{\tau}{\kappa})}{\frac{\tau}{\kappa}}\right)$, that is short and long term are affected in a different way. Therefore, β_1 is called slope parameter.
- The loading attached to β_2 is given by
$$\left(\frac{1-\exp(-\frac{\tau}{\kappa})}{\frac{\tau}{\kappa}} - \exp(-\frac{\tau}{\kappa})\right),$$
that achieves a maximum at mid-term maturities and goes to 0 as τ tends to 0 or $+\infty$. Therefore a change of this parameter affects the curvature of the spot curve. Therefore, β_2 is called convexity parameter.
- κ controls the decay of the loading factors. Large (small) values produce fast (slow) decay and can better fit the curve at short (long) maturities.

Therefore the success of this model is that it is parsimonious having just 4 parameters and it attaches a clear interpretation to them: β_0 is the level, β_1 is the slope and β_2 the convexity parameter.

Svensson model

The Svensson spot rate function extends the Nelson and Siegel model by allowing for more than one local extremum along the maturity profile. This can be useful in improving the fit of yield curves. The spot rate curve is parametrized by 6 parameters
$$\boldsymbol{\theta} = \{\beta_0, \beta_1, \beta_2, \beta_3, \kappa_1, \kappa_2\}$$
and it extends the N&S model by adding a third component, so that the spot curve takes the form
$$R_{sv}(t, t+\tau; \boldsymbol{\theta}) = R_{ns}(t, t+\tau; \{\beta_0, \beta_1, \beta_2, \kappa_1\}) + \beta_3\left(\frac{1-\exp(-\frac{\tau}{\kappa_2})}{\frac{\tau}{\kappa_2}} - \exp(-\frac{\tau}{\kappa_2})\right) \tag{5}$$

No arbitrage models

The web-app performs the term structure fitting using no-arbitrage models, such as the Vasicek [citazione], and the Cox, Ingersoll and Ross [] models. These models, given the risk-neutral dynamics of the instantaneous short rate $r(t), t \leq T$, see Table 1, obtain the discount factor by computing the following expectation $P(t,T) = \tilde{\mathbb{E}}_t\left(e^{-\int_t^T r(s)ds}\right)$. The Vasicek and CIR model assume a mean-reverting process for the short rate r: if it is above (below) its long-run equilibrium value θ, the drift $k(\theta - r)dt$ is negative (positive), driving the rate down (up) toward this long-run value. Clearly, this is possible if the parameter κ is positive. In addition, a larger value of κ implies a faster reversion. The main difference between the Vasicek and CIR model is that in the former the instantaneous variations of the short rate have normal distribution with constant volatility. In the latter model, the diffusion coefficient is proportional to $\sqrt{r(t)}$, implying some heteroskedasticity in the short rate dynamics and at the same time it guarantees the positivity of the short rate. A similar process has been adopted in the Heston model [citazione], to describe the dynamics of the stochastic volatility.

The following Table provides the most important examples of one-factor affine models with constant parameters

Model	Dynamics dr
Vasicek	$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma d\tilde{W}(t)$
CIR	$dr(t) = \kappa\,(\theta - r(t))\,dt + \sigma\sqrt{r(t)}d\tilde{W}(t)$

Table 1: Short rate dynamics in the Vasicek and Cox-Ingersoll-Ross models

The parameter set is
$$\boldsymbol{\theta} = \{\kappa, \theta, \sigma\}.$$
For both models, the spot rate is affine[1] in $r(t)$
$$R\,(t,T;\boldsymbol{\theta}) = -\frac{A\,(t,T;\boldsymbol{\theta})}{T-t} + \frac{B\,(t,T;\boldsymbol{\theta})}{T-t} \times r\,(t)\,,$$
where the functions A and B are
$$\begin{aligned}
B_{vas}\,(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa},\\
A_{vas}\,(t,T) &= \left(B_{vas}\,(t,T) - (T-t)\right)\left(\mu - \frac{\sigma^2}{2\kappa^2}\right) - \frac{\sigma^2 B_{vas}\,(t,T)^2}{4\kappa}\\
B_{cir}\,(t,T) &= \frac{2(e^{\phi_1(T-t)}-1)}{\phi_2\,(e^{\phi_1(T-t)}-1)+2\phi_1},
\end{aligned}$$
where $\phi_1 = \sqrt{\kappa^2 + 2\sigma^2}; \phi_2 = \phi_1 + \kappa$. $A(t,T)$ is given in Brigo-Mercurio, 2006, pag. 64-66.

The long-term spot rate is obtained by letting T to tend to $+\infty$. In Vasicek, it turns out to be
$$R_{vas}\,(t,\infty) = \theta - \frac{\sigma^2}{2\kappa^2},$$
whilst in CIR it is
$$R_{cir}\,(t,\infty) = \frac{2\kappa\theta}{\gamma+\kappa}, \quad \gamma = \sqrt{\kappa^2 + 2\sigma^2}.$$
In the Vasicek model, the term structure of spot rates is

[1] Therefore $P(t,T;\boldsymbol{\theta}) = e^{-B(t,T;\boldsymbol{\theta})r(t+A(t,T;\boldsymbol{\theta}))}$

1. monotonically increasing if $r(t) < R(t, \infty)$;
2. humped if $R(t, \infty) < r(t) < \theta$;
3. monotonically decreasing if $r(t) > \theta$;

In the CIR model, the term structure of spot rates is

1. monotonically increasing if $r(t) < R(t, \infty) - \frac{\sigma^2}{4\kappa^2}$;
2. humped if $R(t, \infty) - \frac{\sigma^2}{4\kappa^2} < r(t) < R(t, \infty) + \frac{\sigma^2}{2\kappa^2}$;
3. monotonically decreasing if $r(t) > R(t, \infty) + \frac{\sigma^2}{2\kappa^2}$;

For both models, the parameter set $\boldsymbol{\theta}$ is augmented by r_t, that is the current value of the short rate is treated as additional parameter. For both models we have the restrictions $\kappa > 0, \sigma > 0$. In addition, for the CIR model we also impose $r_0 > 0, \theta > 0$. In the Vasicek model, it also makes sense to assume $\theta > 0$.

Non parametric fitting

In addition to the use of parametric models, the user can also select non-parametric interpolation. In particular, the web-app allows for

- linear interpolation of spot rates;
- piece-wise constant interpolation of forward rates.

Linear interpolation of spot rates

Given the discount factors for maturities T_1 and T_2 we are interested in the discount factor for maturity $T, T_1 < T < T_2$. We compute the spot rates $R(T, T_2)$ and $R(T, T_2)$ and then we linearly interpolate the T-spot rate

$$R(t, T) = \frac{T_2 - T}{T_2 - T_1} R(T, T_1) + \frac{T - T_1}{T_2 - T_1} R(T, T_2)$$

and then we get the interpolated discount factor

$$P(t, T) = e^{-(T-t)R(t,T)}$$

This procedure allows to reproduce exactly market prices, but gives a very irregular forward curve.

Piece-wise constant interpolation of forward rates

Given the two discount factors, let n to be the number of days between T_1 and T_2. We impose that the daily forward discount factor to be constant and equal to x, so that

$$P(t, T_2) = P(t, T_1) \cdot x^n \tag{6}$$

i.e.

$$x = \left(\frac{P(t, T_1)}{P(t, T_2)} \right)^{\frac{1}{n}}$$

and then we interpolate the T-discount factor using

$$P(t, T) = P(t, T_1) \cdot x^m$$

where m is the number of days between T_1 and T. According to this procedure, the term structure of forward rates with daily tenor turns out to be piecewise constant.

Form field

Two panels (see figure 1) allow the user to chose the model and to upload market data.

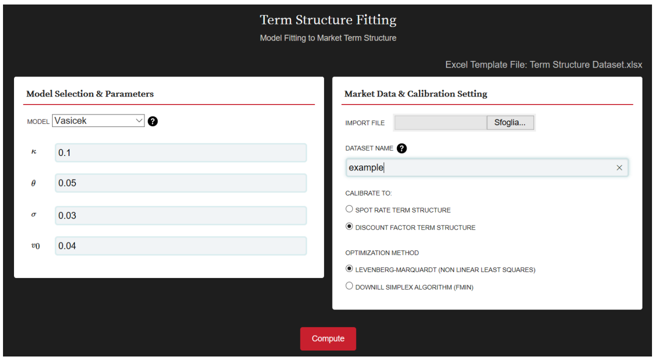


Figure 1: Term-structure input form

Model Selection & Parameters

In this panel, the user can select the model and then he is presented the corresponding parameter set

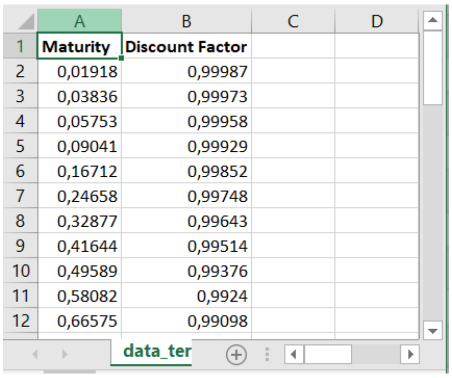
- Nelson Siegel: β_0, β_1, β_2 and κ.
- Svensson: β_0, β_1, β_2, β_3, κ_1 and κ_2.
- Vasicek: κ, θ, σ and r_0.
- CIR: κ, θ, σ and r_0.

The question mark near the field "Model" allows the user to open a pop-up illustrating the main formulas of the selected model.

Once the model has been selected the user must provide the values of the parameters to be used as starting values in the calibration procedure.

Market Data & Calibration Setting

In the "Import File" panel, the user can upload an Excel file. The dataset contained in the file is used to fit the chosen model term structure. The Excel file must comply with a specific template. It's essential that the order and the labels of the columns are as in the template file as illustrated in Figure (2). A template file is downloadable by clicking on "Excel Template File".



	A	B	C	D
1	Maturity	Discount Factor		
2	0,01918	0,99987		
3	0,03836	0,99973		
4	0,05753	0,99958		
5	0,09041	0,99929		
6	0,16712	0,99852		
7	0,24658	0,99748		
8	0,32877	0,99643		
9	0,41644	0,99514		
10	0,49589	0,99376		
11	0,58082	0,9924		
12	0,66575	0,99098		

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Figure 2: Term-structure input form

The two columns in the Excel file represent

- Maturity: time to maturity in years.
- Discount Factor : market zero-coupon prices for each maturity.

In the "Dataset name" field the user can assign a name to the computation session so he can keep track of it in the "Previous simulations" page.

Once the model is flagged and the inputs are imported the user can select between two optimization procedures

- Levenberg-Marquardt (Non Linear Least Squares): the Levenberg-Marquardt algorithm is used.
- Downill simplex algorithm (Fmin): the Downill simplex algorithm is used.

In general, the non-linear least squares performs better so it is the suggested procedure.

In the field "Calibrate to" the user can choose if the model calibration is done using market spot rates, i.e. solving the minimization problem in 2, or discount factors, i.e. solving (3).

The output

The calibration procedure leads to 3 main outputs

1. set of calibrated parameters;
2. calibrated zcb term structure and the corresponding fitting error;
3. calibrated spot rate term structure and the corresponding fitting error.

The first output is the table of optimal parameters for the selected model.

The two main results of the fitting procedure are the model discount factor term structure (3) and the model spot rate term structure with the related fitting error (4).

As case study, the following example takes into account the non linear least squares calibration to to discount factor using the Vasicek and the Svensson model. The dataset is template Excel file (2) downloadable from the webap.

Table of fitted parameters

In Tables 2 and 3 the webapp returns the fitted parameters as well as the root mean square error for discount factors and spot rates. According to the results of these Tables, the Svensson model appears to provided a better fit both in terms of prices (RMSE=0.001) and spot rates (RMSE=0.0002) respect to the Vasicek model (zcb: RMSE=0.0026; rates: RMSE=0.0009).

Table of Fitted Parameters

Model	κ	θ	σ	$v0$	RMSE DF	RMSE SR
Vasicek	0.0809	0.1339	0.0387	0.009	0.0026	0.0009

Table 2: Discount factor term structure fitting Vasicek model

Table of Fitted Parameters

Model	$\beta 0$	$\beta 1$	$\beta 2$	$\beta 3$	$\tau 1$	$\tau 2$	RMSE DF	RMSE SR
Svensson	0.008	-0.0017	0.0125	0.0976	0.4801	10.6139	0.001	0.0002

Table 3: Discount factor term structure fitting Svensson model

The link "Click here to download the fitted term structure" (appearing above the "Table of fitted parameters") allows to download the fitted model discount factors and spot rates for every day starting from the first maturity up to the last one in the dataset.

Discount Factor Term Structure and fitting errors

Figures 3 and 4 provide the following informations: red points are the combinations maturities-market discount factors contained in the template Excel file; the blue line is the fitted model discount factor function. The first figure refers to fitting the Vasicek model, whilst the second is considering the Svensson model. Figures 5 and 6 present the fitting error across the different maturities for the Vasicek and Svensson models. In this comparison, the Svensson model appears to be more accurate than the Vasicek model. A summary is given in Tables 2 and 3 where the root-mean square errors in fitting discount factors and spot rates are given.

Spot Rates Term Structure and fitting errors

The third output is the fitted term structure of spot rates and the corresponding fitting errors. An illustrative example is given in Figure 7.

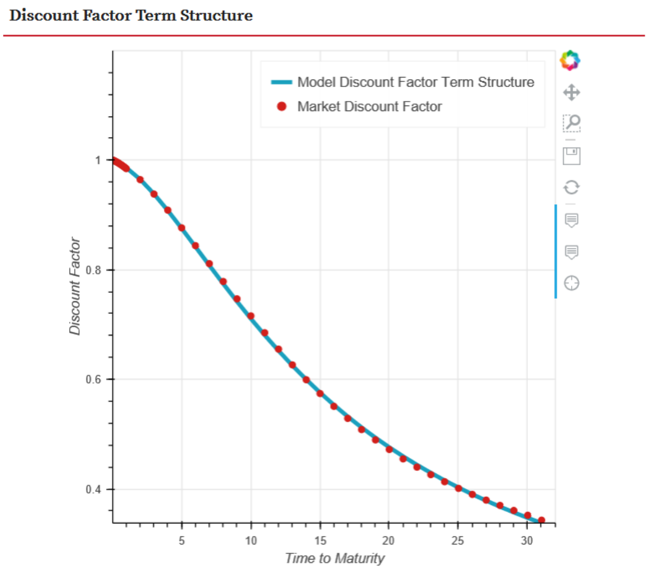

Figure 3: Discount factor term structure fitting Vasicek model

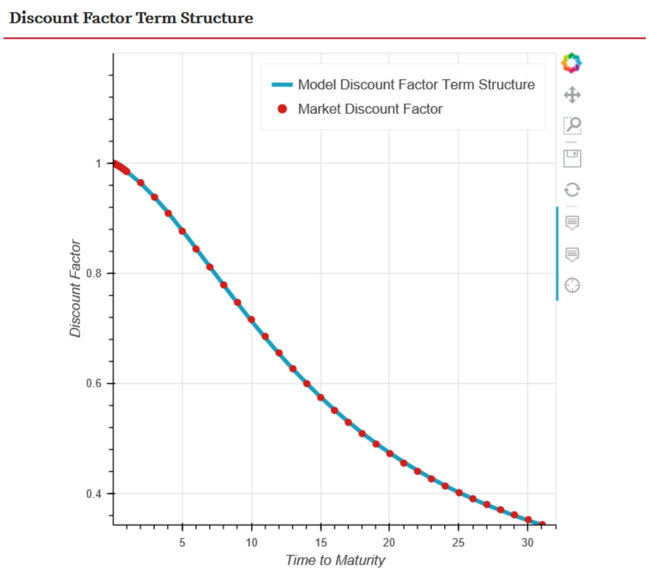

Figure 4: Discount factor term structure fitting Svensson model

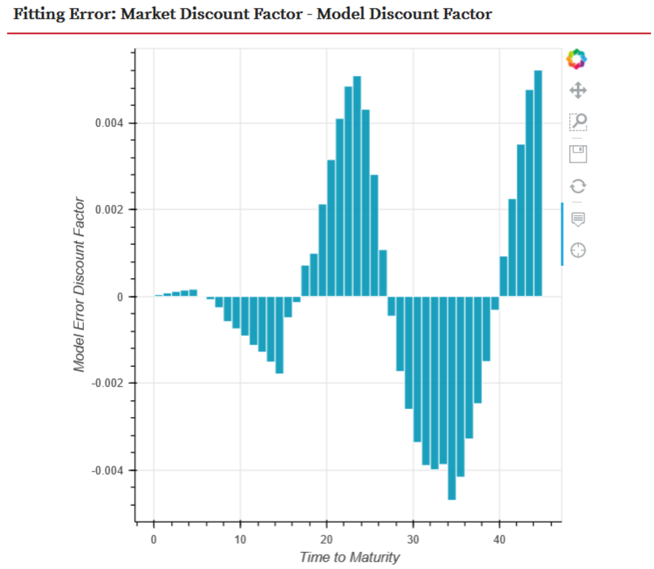

Figure 5: Discount Factor fitting error Vasicek model

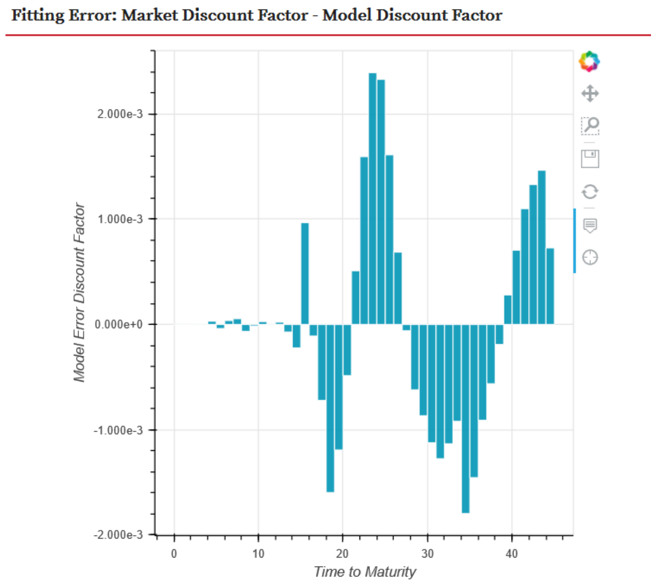

Figure 6: Discount factor fitting error term Svensson model

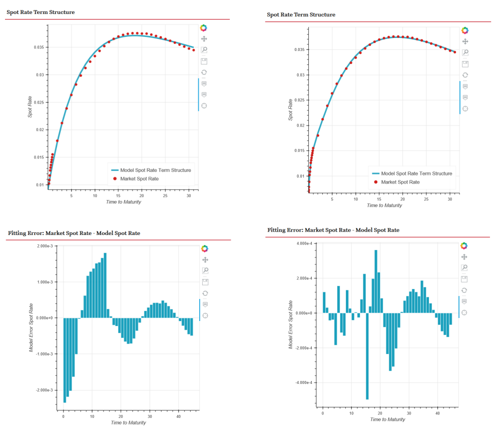

Figure 7: Top Left: fitted spot term structure in the Vasicek model; **Top Right**: fitted spot term structure in the Svensson model; **Bottom Left**: fitting errors in the Vasicek model; **Bottom Right**: fitting error spot term structure in the Svensson model

Chart Tools

In each chart there are interactive tools positioned at the top right. Here are listed all of them starting from the first one.

- ❑ Bokeh Logo: hyperlink to access the library used to create all the interactive graph in the web-application.
- ❑ Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- ❑ Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- ❑ Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- ❑ Reset: The reset tool turns off all the selected tools.
- ❑ Hoover Tool: the hover tool will generate a "tabular" tooltip where each row contains a label, and its associated value.

References

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Svensson, L. E. (1994). *Estimating and interpreting forward interest rates: Sweden 1992-1994* (Tech. Rep.). National bureau of economic research.

Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of financial economics*, *5*(2), 177–188.