FINANCE WEB APP

EUROPEAN OPTION - Lévy Process

Pricing European option with Lévy stochastic processes using Fourier-Cosine series expansion

This tutorial implements the COS method introduced by Fang and Oosterlee (2009) to price European options when the log-returns are modelled by a Lévy process. Option prices are computed numerically by recovering underlying density function from characteristic function using the COS method by Fang and Oosterlee (2009) "A novel pricing method for European option based on Fourier-cosine series expansions". Cos method offers a highly efficient way to . In this section we propose the pricing of European options and four Lévy processes : Normal (Geometric Brownian Motion), VG (Variance Gamma), NIG (Normal Inverse Gaussian) and CGMY. Furthermore given the model option prices we extract the Black-Scholes implied volatilities so we can understand how the different model parameters affect the shape of the volatility smile. Another tutorial covers the pricing of European options using the Heston stochastic volatility model.

Background information

Lévy processes display a number of interesting features. First, they are the most direct generalization of model based on Brownian motion (BM); second, they are analytically tractable; third, they are general enough to include a wide variety of patterns, so that they can account for smile and skew effects occurring in option prices; fourth, the i.i.d. structure of Lévy processes simplifies the estimation of the corresponding parameters under the historical probability measure. Any Lévy process is fully determined by the characteristic function of its log-price increment $z(\Delta)$

$$\psi^{H}(\gamma) = \mathbb{E}_t^*\left(e^{i\gamma z(\Delta)}\right) = e^{m\Delta + \psi_\Delta(\gamma)}, \tag{1}$$

where Δ is the time increment and μ is the drift parameter. Equivalently, we can specify the price process in terms of the characteristic exponent $\psi_\Delta(\gamma)$, which is defined as the logarithm of the characteristic function. In Table 1, we list a few parametric Lévy processes and their associated characteristic exponent. The normal model is a benchmark assumption: we have the purely diffusive Brownian motion, which gives rise to the geometric Brownian motion (GBM) process for the price of the underlying. The remaining models reported in Table 1, are subordinated Brownian motions: in other words, they can be interpreted as Brownian motions subject to a stochastic time change which is related to the level of activity in the market.

Model (Parameters)	$\psi_\Delta(\gamma)$
$G(\sigma)$	$-\frac{\sigma^2}{2}\gamma^2\Delta$
$NIG(\alpha,\beta,\delta)$	$-\delta\Delta\left(\sqrt{\alpha^2-(\beta+i\gamma)^2}-\sqrt{\alpha^2-\beta^2}\right)$
$VG(\sigma,v,\theta)$	$-\frac{\Delta}{v}\ln\left(1-i\theta v\gamma+\left(\sigma^2 v/2\right)\gamma^2\right)$
$CGMY(C,G,M,Y)$	$C\Delta\Gamma(-Y)\left((M-i\gamma)^Y-M^Y+(G+i\gamma)^Y-G^Y\right)$

Table 1: Characteristic exponents of some parametric Lévy processes: G (Gaussian), NIG (Normal Inverse Gaussian), VG (Variance Gamma), CGMY (Carr-Geman-Madan-Yor), M (Meixner), DE (Double Exponential), JD (Jump Diffusion or Merton), S (Stable).

The parameters of the different models must satisfy some constraints, as given in Table 2.

Model	Parameters Restriction
N	$\sigma > 0$
NIG	$\alpha > 0, -\alpha < \beta < \alpha, \delta > 0$
VG	$v > 0, G > 0, M > 0$
CGMY	$C > 0, G > 0, M > 0, Y < 2$

Table 2: Parameter restrictions of some parametric Lévy processes

So far, the drift parameter m in (1) has been left unspecified. Moreover, due to the incompleteness of the market, we have to choose a martingale measure for the risk-neutral pricing of derivatives. In particular, a mathematical tractable choice consists in choosing the value of m such that the stock price discounted by the money-market account is a martingale, *i.e.*. $\mathbb{E}_t^*\left[X(T)/B(T)\right] = X(t)/B(t)$, $\forall T \geq 0$. A simple algebraic manipulation shows that m must be set equal to

$$m = r - \frac{\psi_\Delta(-i)}{\Delta}, \tag{2}$$

where r denotes the constant risk-free rate.

Inverting the characteristic function

Given the characteristic function, if $\int_{\mathbb{R}} |\psi^H(\gamma)| \, d\gamma < \infty$, then the probability density function $f(z(T))$ of the log-price increment can be recovered by the following Fourier inversion formula

$$\frac{1}{\pi} \int_0^{+\infty} \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\gamma z(T)} \psi^H(\gamma)\right) \mathrm{d}u, \tag{3}$$

where $\mathrm{Re}(\cdot)$ stands for the real part of its argument. The COS method approximates the density function via the following Fourier-Cosine expansion

$$f(y) \approx \frac{F_0}{2} + \sum_{k=1}^{\infty} F_k \cos\left(k\pi \frac{y-a}{b-a}\right). \tag{4}$$

where

$$F_k = \frac{2}{b-a} \Re\left(e^{k\pi \frac{-a}{b-a}} \psi^H\left(\frac{k\pi}{b-a}\right)\right)$$

The cumulative distribution function follows by integration, and it is possible to write

$$F(x) = \sum_{k=0}^{N}{}' F_k \psi_k(a, x)$$

where $\sum'$ indicates that the first term is weighted by 0.5 and $\psi_k(c, d)$ equal to

$$\frac{b-a}{k\pi}\left(\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right) \tag{5}$$

if $k \neq 0$ and to $d - c$ elsewhere.

The price of the European call option can be written as

$$C_0 = e^{-rT} K \sum_{k=0}^{N-1}{}' V_k F_k, \tag{6}$$

with

$$V_k = \chi_k(0, b) - \psi_k(0, b),$$

and $\chi_k(c, d) = \beta(d) - \beta(c)$, with $\beta(y)$ given by $\frac{e^y}{1+\left(\frac{k\pi}{b-a}\right)^2}\left(\cos\left(k\pi\frac{y-a}{b-a}\right) + \frac{k\pi}{b-a}\sin\left(k\pi\frac{y-a}{b-a}\right)\right)$.
The accuracy of the overall approximation depends on the approximation error originated by using the original characteristic function instead of its truncation to a subinterval $[a, b]$, and the truncation error originated by considering only the first $N > 0$ terms of the summation.

Form Field

There are 3 different groups (Figure 1) of input parameters. Here we are going to explain the features of each one.

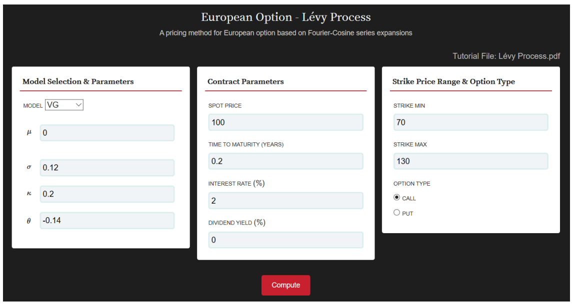

Figure 1: Input form

Model Selection & Parameters

In this subsection are listed the popular Lévy processes and their parameters.

- Normal (Geometric Brownian Motion): μ the mean and σ (annualized).
- VG (Variance Gamma): μ, σ, κ and θ (annualized).
- NIG (Normal Inverse Gaussian): μ, σ, κ and θ (annualized).
- CGMY: μ, C, G, M and Y (annualized).

Contract Parameters

- Spot price: the price of underlying asset at time 0.
- Time to maturity (years): expiring date of the European option.
- Interest rate (%): the risk free rate (annual continuously compounded).
- Dividend yield (%): the dividend yield paid by the underlying asset (annual continuously compounded).

Strike Price Range & Option Type

- Strike Min - Strike Max: it's the range in which are computed European option prices and implied volatilities.
- Option type: A choice between call or put European option prices.

The output

The procedure implemented in this section leads the user to 3 main results. Figure 2 presents a comparison between the recovered density function in case of VG model (blue line) and the benchmark normal (red line) with mean and variance in Table 3. In Figures 4 and 5 is displayed the implied volatility profile for different strikes.
Furthermore we have a pop-up table containing the

prices of European option estimated with the COS method for different strikes (Table 5). The range of the "Implied Volatility Profile" (Figure 4) and the "Table of Volatility & Prices" (Table 5) depends on the value of "Strike min" and "Strike max" chosen by the user in the panel "Strike Price Range & Option Type". Next we present a comparison between the output of VG model and Normal model.

Recovered density function of Lévy Processes

In Figure 2 the blue line is the recovered density function of VG model computed using the COS method. The red line, the benchmark normal, is the distribution followed by the underlying log returns in normal case with mean and variance of the selected Lévy process (Table 3). Whenever the user chooses the "Normal" ("Model Selection & Parameters" Figure 1) the plots overlap each other (Figure 3) since log returns are normally distributed.

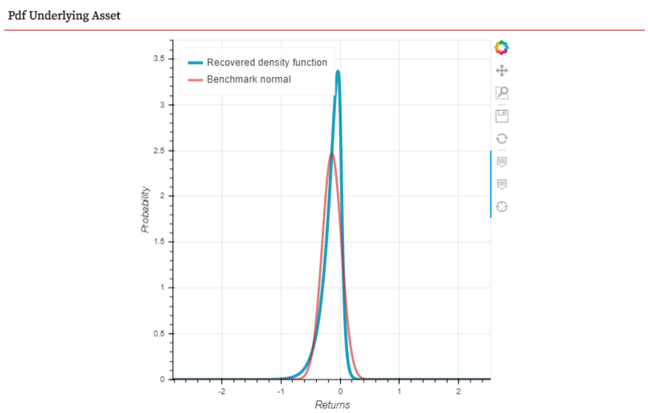

Figure 2: Recovered density function of VG model vs Benchmark (input in Figure 1) Normal

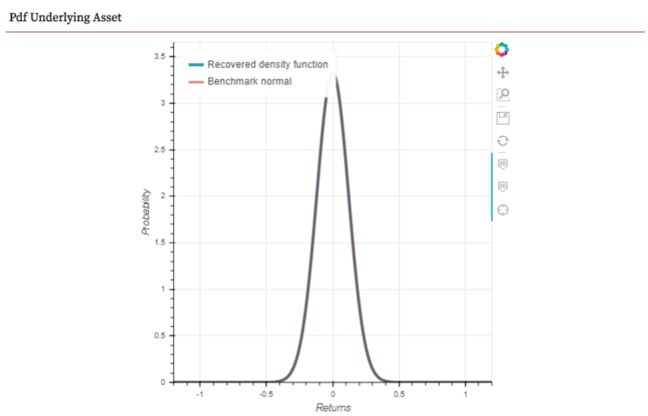

Figure 3: Recovered density function of Normal case (μ = 0, σ = 0,12) vs Benchmark normal

Table of Moments

The table displays the moments of the recovered density function. In Table 3 we find that recovered density function of VG model is negatively skewed and more leptokurtic than the benchmark normal.

Table of Moments

Model	Mean	Variance	Skewness	Kurtosis
VG	-0.14	0.0262	-0.6761	8.2999

Table 3: Table of Moments VG model

Table of Moments

Model	Mean	Variance	Skewness	Kurtosis
Normal	0.0	0.0144	0	3

Table 4: Table of Moments Normal case

Implied Volatility Profile

In this step implied volatilities are computed starting from the European option prices obtained with the COS method. The procedure implemented is a root-finding algorithm the Brent-Dekker method applied on Black-Scholes formula. The output by selecting "VG" with the parameters in "Model Parameters" (Figure 1)is a negatively sloped implied volatility profile (Figure 4), implying a negative skewness (Table 3). The user can also represent the implied volatility profile under normality assumption using the geometric Brownian motion process. In this case the resulting implied volatility is flat (Figure 5), the volatility is the same for each strike.

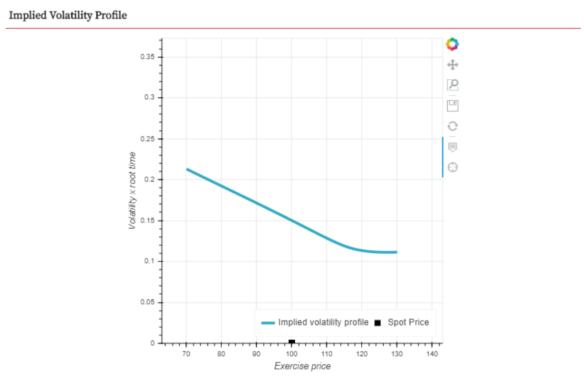

Figure 4: Implied volatility profile VG model

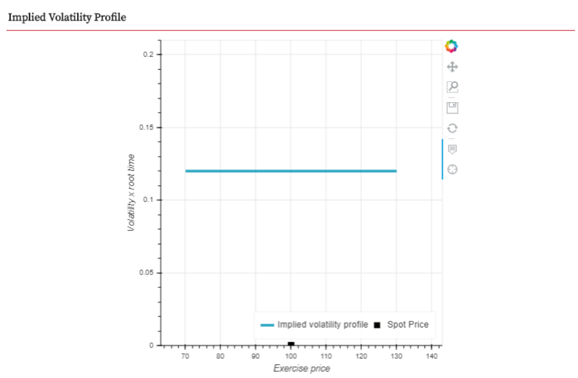

Figure 5: Implied volatility profile Normal case model μ = 0, σ = 0.12

Table of results

The user can click on "View Details", the output is a pop up table with the columns listed here.

- ❑ Strike: the strikes generated in a range between "Strike Min" and "Strike Max".
- ❑ Prices : the prices of the European Option estimated with COS method.
- ❑ Implied Volatility : the implied volatilities computed with Brent-Dekker method via Black and Scholes.

Table of Volatility & Prices

Strike	Prices	Implied Volatility
70.0	33.055059	0.32748
71.0	32.194073	0.32495
72.0	31.339534	0.322431
73.0	30.491673	0.319922
74.0	29.650724	0.317422
75.0	28.816921	0.314932
76.0	27.990504	0.31245
77.0	27.171715	0.309976
78.0	26.360794	0.307509
79.0	25.55799	0.305048
80.0	24.76355	0.302593
81.0	23.97772	0.300144
82.0	23.200754	0.297699
83.0	22.432902	0.295258

Export Table

Table 5: Table of results in VG model

	A	B	C
1	Strike	Prices	Implied Volatility
2	70	33.05506	0.32748
3	71	32.19407	0.32495
4	72	31.33953	0.322431
5	73	30.49167	0.319922
6	74	29.65072	0.317422
7	75	28.81692	0.314932
8	76	27.9905	0.31245

data

Figure 6: Table of results of VG model saved in an Excel file

Understanding the role of the model parameters

We leave to the user to experiment the web app and try to understand the role of the different parameters on the moments, the shape of the density function and the implied volatility profile.

Chart Tools

In each chart there are interactive tools positioned at the top right. Here are listed all of them starting from the first one.

- ❑ Bokeh Logo: hyperlink to access the library that we used to create all interactive graph in the web-application.
- ❑ Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- ❑ Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- ❑ Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- ❑ Reset: The reset tool turns off all the selected tools.
- ❑ Hoover Tool: the hover tool will generate a "tabular" tooltip where each row contains a label, and its associated value.

References

Fang, F., & Oosterlee, C. W. (2009). A novel pricing method for european options based on fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, *31*(2), 826–848.