FINANCE WEB APP

EUROPEAN OPTION - Heston model

Pricing European option in the Heston stochastic volatility model using Fourier-Cosine series expansion

This tutorial implements the COS method introduced by Fang and Oosterlee (2009) to price European options in the Heston stochastic volatility model Heston (1993). Furthermore given the model option prices we extract the Black-Scholes implied volatilities so we can understand how the different model parameters affect the shape of the volatility smile.

Another tutorial covers the pricing of European options using different Lévy processes.

Background information

The Heston stochastic volatility model is described by two coupled stochastic differential equations, describing the dynamics of the stock price and of its instantaneous variance. The price dynamics is given by

$$dP(t) = (r-q)P(t)dt + \sqrt{v(t)}P(t)dW_1(t), \quad P(0) = p_0$$

where p_0 is the initial stock price, r is the (constant) instantaneous risk-free rate, q is the (constant) dividend yield, and $v(t)$ is the instantaneous variance whose dynamics is described by the **mean-reverting square-root** process

$$dv(t) = \lambda(\hat{v} - v(t))\,dt + \chi\sqrt{v(t)}dW_2(t), \quad v(0) = v_0$$

The Brownian motions entering the two stochastic differential equations have instantaneous correlation ρdt

$$\mathbb{E}(dW_1(t)dW_2(t)) = \rho dt.$$

The parameters restrictions are

$$P_0, v_0, k, \hat{v}, \chi > 0, \quad \rho \in [-1, 1]$$

The variance process is characterized by mean-reversion. The expected change in the variance is given by

$$\mathbb{E}(dv(t)) = k(\theta - v(t))\,dt$$

and it is positive or negative depending on the current level of the variance with respect to the long-term value $\hat{v}$. Given that $\lambda > 0$, if $v(t) > \hat{v}$, the drift is negative so we expect a reduction in the variance level. Viceversa, if $v(t) < \hat{v}$. The size of expected change depends also on the parameter χ, that is called speed of mean-reversion: greater this parameter, larger the expected change and therefore faster the movement of the variance towards the long term value $\hat{v}$. The square-root appearing in the diffusion component is chosen for two reasons. Indeed, it guarantees that a) the variance remains always non-negative[1], b) the model is analytically tractable, in the sense that the characteristic function of the log-price is known in closed form. The parameter χ is called volatility of variance (or vol-vol parameter in short).

The key model parameters are the correlation parameter ρ and the vol-vol parameter χ. Indeed, negative correlation between the return and the volatility process is needed in order to generate an asymmetric distribution. A higher value of the vol-vol parameter generates a higher kurtosis in the log-price distribution. This is reflected in the shape of the volatility skew, i.e. the plot of implied volatilities versus option strikes. The correlation coefficient controls the slope of the skew: if $\rho < 0$ ($\rho < 0$) the skew is negatively (positively) sloped. The vol-vol parameter controls the kurtosis of the distribution.

The density function of log-prices is not known in closed form, but only via the characteristic function that is available in closed form:

$$\psi^H(u) := \psi^H(u; p(t), v(t), T-t) = \quad (1)$$

$$\mathbb{E}_t\left[e^{iuln(P(T))}\right] = e^{C(u,T-t)+D(u,T-t)v_t+iu\ln(P(t))}, \quad (2)$$

where the functions C and D are given by in terms of $\tau = T - t$

$$C(u,\tau) = iu(r-q)\tau + \frac{\lambda\hat{v}}{\chi^2}\left((\lambda - i\rho\chi u - d)\,\tau - 2\ln\frac{1-ge^{-d\tau}}{1-g}\right),$$

[1] The variance process is always positive and cannot reach 0, if the so called *(Feller condition)* $\chi^2 \leq 2\lambda\hat{v}$ is satisfied.

$$D(u,\tau) = \frac{1}{\chi^2}\left(\lambda - i\rho\chi u - d\right)\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}},$$

with $i = \sqrt{-1}$, $\tau = T - t$ and

$$\begin{aligned} d &= \sqrt{(i\rho\chi u - \lambda)^2 + \chi^2\left(iu + u^2\right)}, \\ g &= \frac{\lambda - i\rho\chi u - d}{\lambda - i\rho\chi u + d}. \end{aligned}$$

Given the characteristic function, if $\int_{\mathbb{R}} |\psi^H(u)|\, du < \infty$, then the probability density function $f(\log(P(T)))$ of the log-price can be recovered by the following Fourier inversion formula

$$\frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}u\log(P(T))}\psi^H(u)\right) \mathrm{d}u, \tag{3}$$

where $\mathrm{Re}(\cdot)$ stands for the real part of its argument. The COS method approximates the density function via the following Fourier-Cosine expansion

$$f(y) \approx \frac{F_0}{2} + \sum_{k=1}^{\infty} F_k \cos\left(k\pi\frac{y-a}{b-a}\right). \tag{4}$$

where

$$F_k = \frac{2}{b-a}\Re\left(e^{k\pi\frac{-a}{b-a}}\psi^H\left(\frac{k\pi}{b-a}\right)\right)$$

The cumulative distribution function follows by integration, and it is possible to write

$$F(x) = {\sum_{k=0}^{N}}' F_k \psi_k(a,x)$$

where $\sum'$ indicates that the first term is weighted by 0.5 and $\psi_k(c,d)$ equal to

$$\frac{b-a}{k\pi}\left(\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right) \tag{5}$$

if $k \neq 0$ and to $d - c$ elsewhere.

The price of the European call option can be written as

$$C_0 = e^{-rT} K {\sum_{k=0}^{N-1}}' V_k F_k, \tag{6}$$

with

$$V_k = \chi_k(0,b) - \psi_k(0,b),$$

and $\chi_k(c,d) = \beta(d) - \beta(c)$, with $\beta(y)$ given by $\frac{e^y}{1+\left(\frac{k\pi}{b-a}\right)^2}\left(\cos\left(k\pi\frac{y-a}{b-a}\right) + \frac{k\pi}{b-a}\sin\left(k\pi\frac{y-a}{b-a}\right)\right)$.

The accuracy of the overall approximation depends on the approximation error originated by using the original characteristic function instead of its truncation to a subinterval $[a,b]$, and the truncation error originated by considering only the first $N > 0$ terms of the summation.

Form Field

Input parameters are organized in 3 panels (Figure 1).

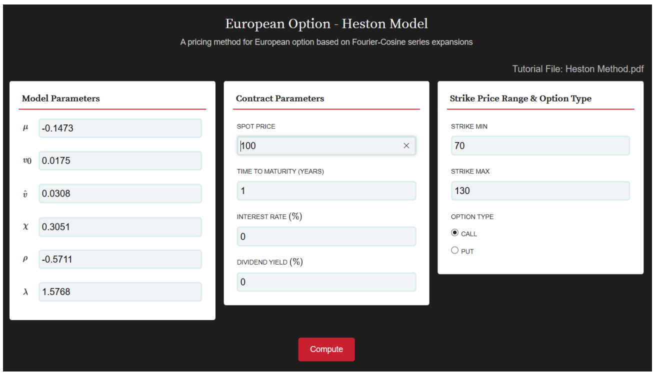

Figure 1: Input form

Model Parameters

In this panel, we can insert the Heston model parameters.

- ❑ v_0 : instantaneous variance.
- ❑ $\hat{v}$: mean level of variance - long run variance.
- ❑ ρ : correlation between the variance and log-price increments.
- ❑ χ : volatility of volatility.
- ❑ λ : variance mean reversion speed parameter.

Underlying & Contract Parameters

In this panel, the user can insert the parameters of the risk-neutral price process (initial stock price, risk-free rate, dividend yield) and of the contract (time to maturity).

- ❑ Spot price: the initial price p_0 of underlying asset at time 0.
- ❑ Time to maturity (years): expiring date of the European option.
- ❑ Interest rate (%): the risk free rate r (annual continuously compounded).
- ❑ Dividend yield (%): the dividend yield q paid by the underlying asset (annual continuously compounded).

Strike Price Range & Option Type

In this panel, the user can input the range of strike prices to be used in pricing the European option and the option type (call or put).

- ❑ Strike Min - Strike Max: it's the range in which are computed European option prices and implied volatilities.

- ❑ Option type: A choice between call or put European option prices.

The output

The procedure implemented in this app allows the user to analyze the main features of the Heston model, by generating: the probability density function of the model, see Figures 2 - 3 - 4; the Table of moments (mean, variance, skewness and kurtosis) of the log-price process, see Tables 1 - 2 - 3; the implied volatility smile generated by the model, see Figures 5 - 6 - 7 . Furthermore, below the Table of moments, we have a button (View Details) that allows the user to create a pop-up table containing the prices of European option with strikes falling in the interval "Strike min"-"Strike max" chosen by the user in the panel "Strike Price Range & Option Type". In the following, we illustrate how the web app can help the user to understand how the correlation parameter ρ affects the shape of the recovered distribution, the moments and the shape of the implied volatility profile. To do this we assign to ρ three values: -0.8 (negative correlation between log-price and variance increments), 0 (independence), 0.8 (positive correlation).

Recovered density function of Heston Model

In Figures 2 - 3 - 4 we compare the density function of returns in the Heston model (blue line) with the Gaussian benchmark (red curve), i.e. the normal PDF having same mean and variance as in the Heston model. This comparison highlights how the Heston model can generate a distribution having excess of kurtosis and positive or negative skewness.

- ❑ Left-skewed distribution ($\rho = -0.8$): the recovered implied distribution is negatively skewed. That's is the effect of a negative value of Figure 2.

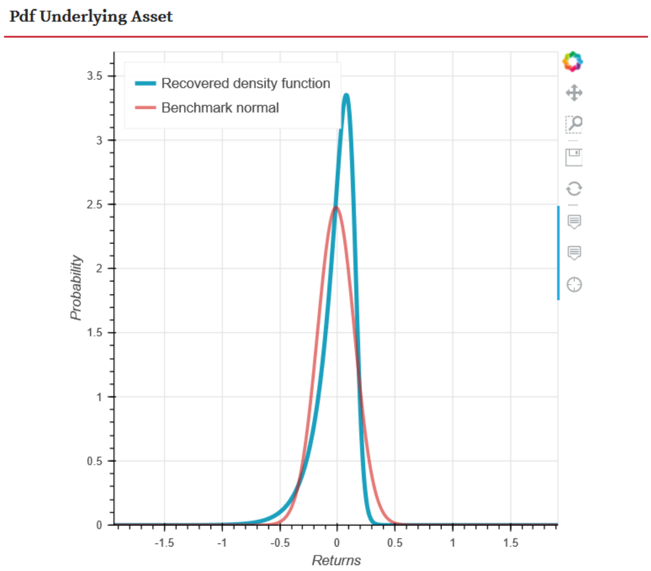

Figure 2: Recovered density heston model ($\rho = -0.8$)

- ❑ Skew normal distribution ($\rho = 0$): the recovered implied distribution is symmetric but it has an excess of kurtosis with respect to the benchmark normal distribution. That's the effect of a ρ close to zero (Figure 3) but a positive value of the vol-vol parameter.

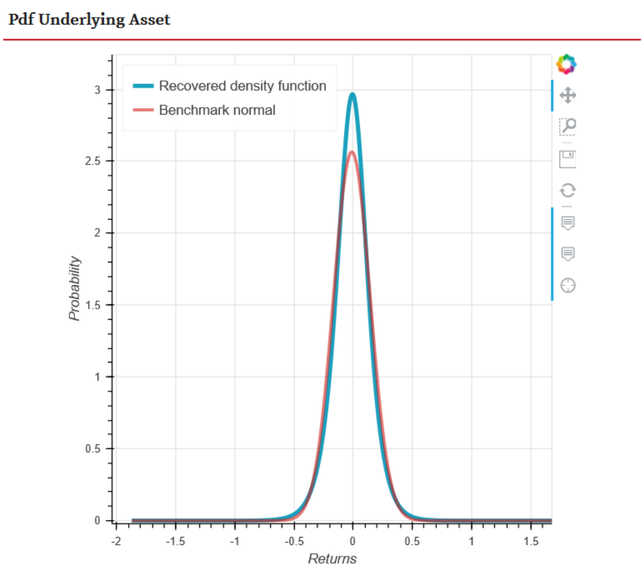

Figure 3: Recovered density Heston model ($\rho = 0$)

- ❑ Right-skewed distribution ($\rho = 0.8$): the recovered implied distribution is positively skewed. That's the effect of a positive value of ρ (Figure 4).

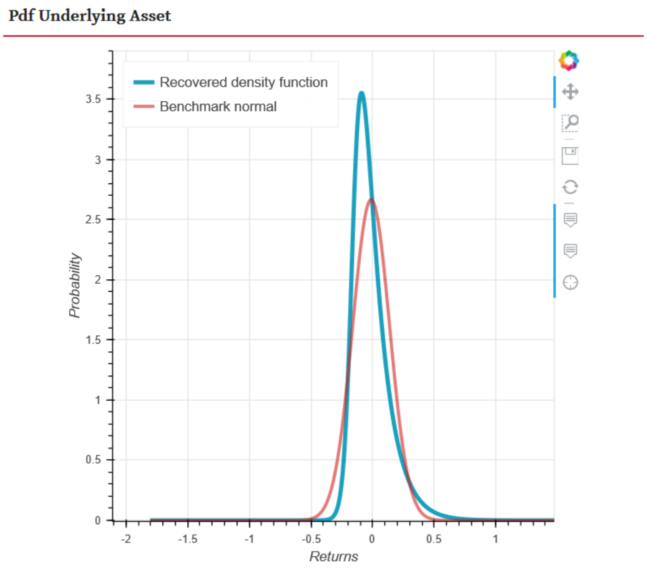

Figure 4: Recovered density Heston model ($\rho = 0.8$)

Table of Moments

Tables 1 - 2 - 3 display the moments of the recovered density function of Heston model for three different values of the correlation parameter.

❑ Negative skewness ($\rho = -0.8$).

Table of Moments

Mean	Variance	Skewness	Kurtosis
-0.0121	0.0259	-1.436	6.5145

Table 1: Table of Moments Heston model ($\rho = -0.8$)

❑ Skewness close to zero ($\rho = 0$).

Table of Moments

Mean	Variance	Skewness	Kurtosis
-0.0121	0.0242	-0.0985	4.2807

Table 2: Table of Moments Heston model ($\rho = 0$)

❑ Positive skewness ($\rho = 0.8$).

Table of Moments

Mean	Variance	Skewness	Kurtosis
-0.0121	0.0224	1.378	6.3641

Table 3: Table of Moments Heston model ($\rho = 0.8$)

Implied Volatilty Profile

Implied volatilities are computed by searching for that value of the volatility of the log-price which, when used as input in the Black-Scholes pricing model, will return a theoretical value equal to the corresponding European option prices given by the Heston model. The calculation of the implied volatility requires a root-solving algorithm that is implemented via Brent-Dekker method in Python.

❑ Negatively sloped volatility profile ($\rho = -0.8$): the implied volatility profile is negatively sloped, implying a negative skew distribution of the underlying log-price.

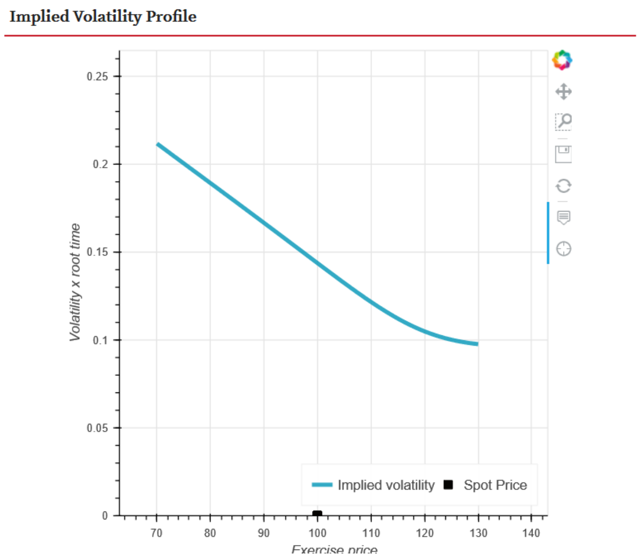

Figure 5: Implied Volatility Profile ($\rho = -0.8$)

❑ Symmetric volatility profile ($\rho = 0$): the implied volatility profile is symmetric because the density function is symmetric, but still exhibits a kurtosis larger than the Gaussian benchmark of 3.

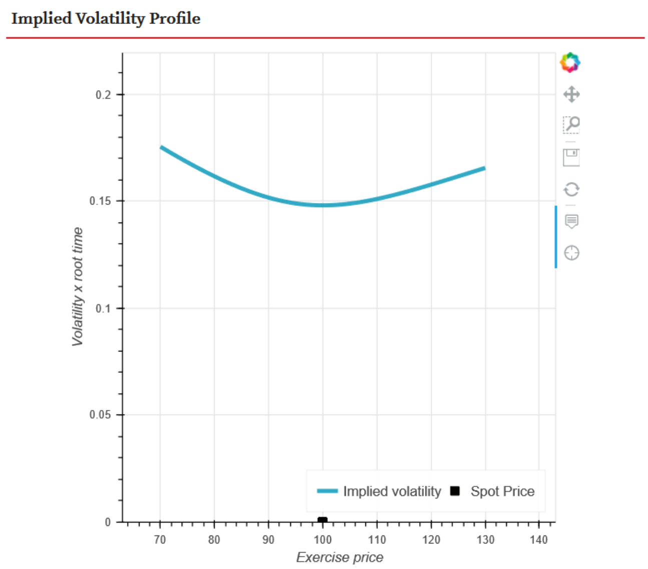

Figure 6: Implied Volatility Profile ($\rho = 0$)

❑ Positively sloped volatility profile ($\rho = 0.8$): the implied volatility profile is positively sloped, implying a positively skewed distribution

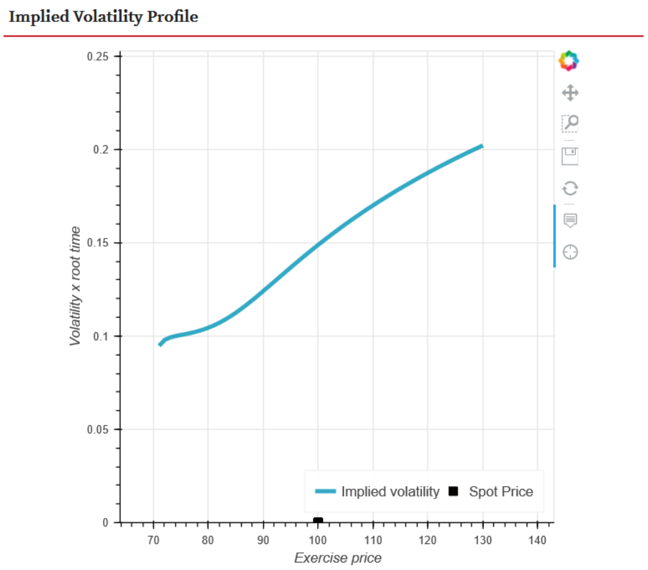

Figure 7: Implied Volatility Profile ($\rho = 0.8$)

Table of Volatility and Price

The user can click on "View Details". The output is a pop up table with the columns as follows

- ❑ Strike: the strikes generated in a range between "Strike Min" and "Strike Max".
- ❑ Prices : the prices of the European Option estimated with COS method.
- ❑ Implied Volatility : the Black and Scholes implied volatilities.

Table of Volatility & Prices ×

Strike	Prices	Implied Volatility
50.0	50.043926	0.281119
51.0	49.051045	0.278614
52.0	48.059137	0.276134
53.0	47.06831	0.273679
54.0	46.078684	0.271249
55.0	45.090388	0.268843
56.0	44.103561	0.266459
57.0	43.118354	0.264097
58.0	42.134931	0.261756
59.0	41.153469	0.259436
60.0	40.174156	0.257137
61.0	39.197197	0.254857
62.0	38.222813	0.252596
63.0	37.251237	0.250355
64.0	36.282723	0.248132

Export Table

Table 4: Table of results in Heston model

	A	B	C
1	Strike	Prices	Implied Volatility
2	50	50,01467	0,250034
3	51	49,01786	0,248151
4	52	48,02165	0,24629
5	53	47,02612	0,244449
6	54	46,03139	0,242627
7	55	45,03756	0,240825
8	56	44,04477	0,239041

data

Figure 8: Table of results in Heston model stored in an Excel file

Understanding the role of the model parameters

We leave to the user to experiment the web app and try to understand the role of the different parameters on the moments, the shape of the density function and the implied volatility profile.

Chart Tools

In each chart there are interactive tools positioned at the top right. Here are listed all of them starting from the first one.

- ❑ Bokeh Logo: hyperlink to access the library that we used to create all interactive graph in the web-application.
- ❑ Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- ❑ Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- ❑ Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- ❑ Reset: The reset tool turns off all the selected tools.
- ❑ Hoover Tool: the hover tool will generate a “tabular” tooltip where each row contains a label, and its associated value.

References

Fang, F., & Oosterlee, C. W. (2009). A novel pricing method for european options based on fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, *31*(2), 826–848.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, *6*(2), 327–343.