FINANCE WEB APP
IMPLIED DISTRIBUTION

Market Application

THIS section regards a method implemented by Shimko (1993) "Bounds of probability". This paper explains how European option prices can be used to determine the probability that future index values will lie in a given range, enabling the user to design better speculative strategies and calculate better hedge ratios. The fact that probability distributions can be recovered from option prices was first discovered by Breeden and Litzenberger (1978). In this case Shimko (1993) explains how the original Breeden and Litzenberger (1978) approach can be implemented nonparametrically to recover risk neutral implied probability of future index values. The entire probability distribution gives a complete set of information of underlying, instead of Black-Scholes, it can provide a measure of the skewness of a distribution and it's kurtosis. For example market may place relatively greater probability on downward price movement than upward movement is known as negative skew. When traders speak of "trading the skew" they are usually referring to predicting the slope of the implied volatility curve, and choosing an option position that profits if their forecast materialises. The use of the word "skew" is statistically correct, since a negatively sloped implied volatility curve gives rise to a negatively skewed probability distribution index. The skewness implied by the Black-Scholes model is uniformly small and positive.

Background information

The inverse problem consisting of the identification of a risk-neutral distribution $q_{t+\tau}$ implied by option prices was first addressed in a seminal paper by Breeden and Litzenberger (1978). These authors show that the risk-neutral density $q_{t+\tau}$ is recovered from option prices as

$$q_{t+\tau}(x) = e^{r\tau} \left.\frac{\partial^2 c_t(K,\tau)}{\partial K^2}\right|_{K=x}. \tag{1}$$

Implementing this formula requires the knowledge of option prices for a continuum of strikes. Of course this is not possible in practice and infinitely many density functions are compatible to any given set of option prices over a finite range of strikes. However, some basic constraints have to be satisfied when constructing a risk-neutral density. For example, a well-defined risk neutral density is non-negative, integrates to one, and prices exactly all calls and puts. In Shimko (1993) is proposed to approximate the implied volatility curve as function of the strikes, $\sigma(K)$ to recover a continuous formula for the call prices. Once, we have this interpolation function, we can replace in the Black-Scholes formula and recover the probability distribution through the first two derivatives of the pricing formula wrt to the strike price:

$$\frac{\partial c(K,\sigma(K))}{\partial K} = \frac{\partial c}{\partial K} + \frac{\partial c}{\partial \sigma}\frac{\partial \sigma(K)}{\partial K}, \tag{2}$$

$$\begin{aligned}\frac{\partial^2 c(K,\sigma(K))}{\partial K^2} &= \frac{\partial^2 c}{\partial K^2} + 2\frac{\partial^2 c}{\partial \sigma \partial K}\frac{\partial \sigma(K)}{\partial K} \\ &+ \frac{\partial c}{\partial \sigma}\frac{\partial^2 \sigma(K)}{\partial K^2} + \frac{\partial^2 c}{\partial \sigma^2}\left(\frac{\partial \sigma(K)}{\partial K}\right)^2\end{aligned} \tag{3}$$

In these expressions we can compute:

1. $\frac{\partial^2 c}{\partial K^2}$, $\frac{\partial^2 c}{\partial \sigma \partial K}$, $\frac{\partial c}{\partial \sigma}$ and $\frac{\partial^2 c}{\partial \sigma^2}$ using the Black-Scholes formula.
2. $\frac{\partial \sigma(K)}{\partial K}$ and $\frac{\partial^2 \sigma(K)}{\partial K^2}$ from the interpolating function of the volatility curve.

Clearly, the estimate of the risk neutral probability distribution is very sensitive to the changes in the volatility so we should look for an interpolating function that is as smooth as possible, but which still fits the observed option values closely enough.

The main disadvantage of this approach is that it does not recover the tails of the risk-neutral density function outside the range of available strike prices.

Therefore some extrapolating procedure has to be adopted (**Step 3**).

- ❑ **Step 1**: Use a quadratic interpolation for the implied volatility

$$\sigma(K) = a + bK + cK^2$$

with the coefficients that can be estimated by OLS.

- **Step 2**: Exploiting the BS formula, we have ($\tau = T - t$).

  1.
  $$e^{-r\tau}\frac{\partial c\left(K,\sigma\left(K\right)\right)}{\partial K} = 1 - \mathcal{N}\left(d_2\right) + K\phi\left(d_2\right)\left(b + 2cK\right) \tag{4}$$

  2.
  $$e^{-r\tau}\frac{\partial^2 c(K,\sigma(K))}{\partial K^2} = -n\left(d_2\right)\left(d_{2K} - (b+2cK)(1 - d_2 d_{2K}) - 2cK\right) \tag{5}$$

  where $d_{1,2} = (\ln(SD/KB) \pm 0.5\sigma^2(K))/\sigma(K)$ and $d_{1K} = -\frac{1}{K\sigma(K)} + (1 - \frac{d_1}{\sigma(K)})(b+2cK)$, $d_{2K} = d_{1K} - (b+2cK)$.

- **Step 3**: Outside the quoted range strike, the implied volatility curve is assumed to be constant and the implied density is taken to be lognormal.

Recovering risk neutral density: "Illustration" vs "Market application"

In the section called "Illustration" the user supplies the value of 3 of volatilities and strikes in order to manage the shape of the implied volatility profile and the impact on implied risk neutral density function. In this section the user can import an Excel file containing a series of call/put market option prices and the implied volatilities are computed by the web app.

Form Field

There are 3 different groups (Figure 1) of input parameters. Here we are going to explain the features of each one.

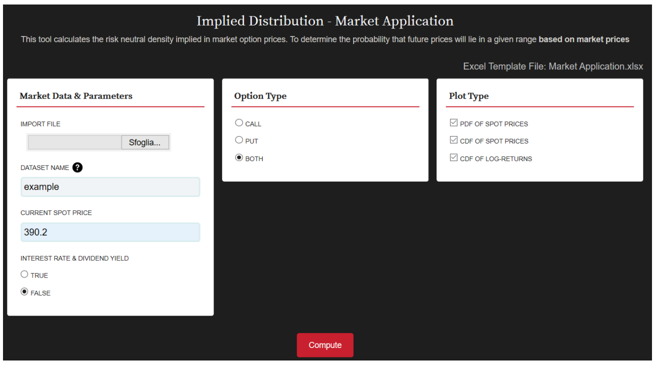

Figure 1: Input form

Contract Parameters

- Import file: the user must select an Excel file containing the relevant inputs (Figure 2) .

- Current spot price: the price of the underlying index observed today on the market.

- Dataset name: the user can assign a name to the computation in order to keep track of the simulation in the page "Previous Simulations".

- Interest rate and divided yield (continuously compounded): if the user flags "False" the dividend yield and risk free rates are computed though call put parity. The option "True" allows the user to insert them as additional inputs.

	A	B	C	D	E
1	Strike	Call_Price	Put_Price	Deselect Row	Time
2	325	66.5	0.3125	1	0.166666667
3	345	46	0.875	0	
4	360	33	2	1	
5	365	27.75	2.625	1	
6	375	20.175	4.25	1	
7	385	13.5	7.125	0	
8	390	9.625	8.75	0	
9	395	7.25	11	1	
10	400	5.375	13.75	1	
11	405	3.375	17	1	
12	410	1.875	19.75	1	
13	425	0.25	34	1	

Figure 2: Template Excel file

Import File

The Excel file containing the market data must have the following structure:

- Strike: strike related to call/put prices.

- Call Price: market European call option prices.

- Put Price: market European put option prices.

- Deselect Column: value equal to 1 (0) allows to include (exclude) data in the computation.

- Time: time to expiry (in years) of the European option.

  A template file is available clicking on "Excel template file".

Option Type

- Call: call prices are used for the fitting of implied volatilities.

- Put: put prices are used for the fitting of implied volatilities.

- Both: out the money call and put prices are used to determine implied volatilities.

Plot Type

In this panel the user can select which plot will be displayed.

- ❑ PDF of spot prices: probability distribution of underlying asset prices.
- ❑ CDF of spot prices: cumulative distribution of underlying asset prices.
- ❑ CDF of log-returns: cumulative distribution of underlying asset log-returns.

  The probability distribution of log-returns is computed by default whenever the user clicks on "Compute".

The output

The procedure implemented in this section leads the user to two main results. The first one is the implied volatility profile represented by a parabola (Figure 3). The second one is a series of graphs and information related to the implied risk-neutral probability distribution (Figures 4 - 5 and Table 2).

Implied volatility profile

The implied volatility profile is determined in **Step 1.** We allowed the implied volatility function to be a parabolic function of the exercise price ($a_0 + a_1 k + a_2 k^2$). On x-axis we have strikes (k) and on y-axis smoothed implied volatilities (Figure 3). In this case, the implied volatility profile is negatively sloped, implying a negative skew distribution of future index values: the market is pricing relatively greater on a fall in the index than on a rise.

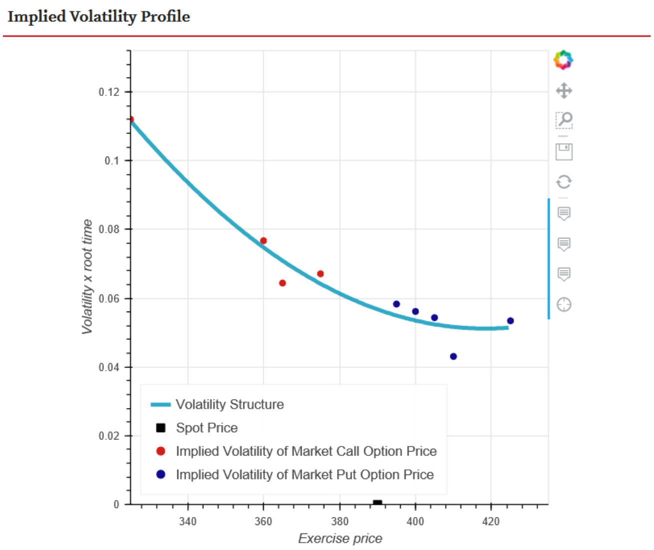

Figure 3: Implied Volatility Profile

In the Table 1 there are three parameters (a_0, a_1, a_2) that control the shape of the implied volatility parabola.

- ❑ a_0 : with higher (lower) a_0 the parabola moves up (down) so higher (lower) implied volatilities.
- ❑ a_1 : positive (negative) a_1 causes the parabola to have a positive (negative) slope.
- ❑ a_2 : positive (negative) a_2 causes the parabola to be concave (convex)
- ❑ R^2 : the goodness-of-fit of the parabolic function (blue line in Figure 3) vs market implied volatilities (blue and red dots in Figure 3).

.

Results

FITTED VOLATILITY SMILE

$a_0 + a_1 \cdot k + a_2 \cdot k^2$

a_0: 1.26338272

a_1: -0.0057944

a_2: 6.9245e-06

R^2: 0.9519117

View Details

Table 1: Results

Recovering risk neutral implied probability distribution from option prices

The fitted implied volatilities can be used to find smoothed call/put prices (via Black-Scholes) and the smoothed call/put prices can be differentiated to find the values of the density function and cumulative distribution for each possible index value at a certain maturity (**Step 2**). The benchmark lognormal (normal) distribution is determined with the same mean and variance as the implied index (log-returns) distribution.

- ❑ Log-returns (Figure 4): on the left (right) side the blue line is the recovered CDF (PDF) of log-returns and the red line is the benchmark lognormal CDF (PDF).

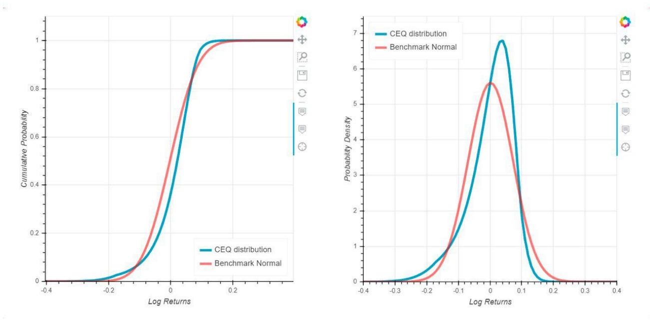

Figure 4: Implied recovered log returns distributions

- Index values (Figure 5): on the left(right) side the blue line is the recovered PDF(CDF) and the red line is the benchmark normal PDF(CDF).

.

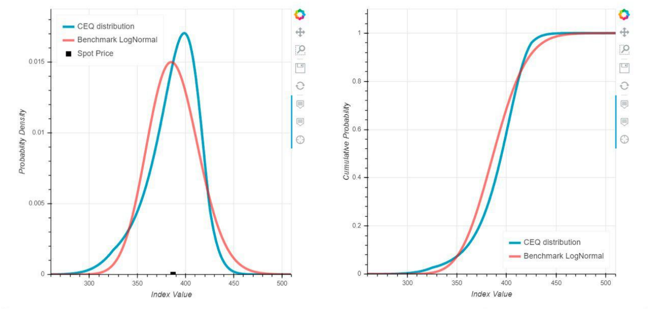

Figure 5: Implied recovered index distributions

Table of moments

There is the possibility for the user to click on "View details" in Table 1. The output is a pop-up table (Table 2) containing the moments of each distribution. In the table we find that the implied index distribution is negatively skewed and more leptokurtic than the benchmark lognormal distribution. This confirm the previous observation from the implied volatility profile on skewness and kurtosis An additional feature is the "Export Table" (Table2) that stores the table in an Excel file that the user can download.

Table of moments ×

Description	Ceq	LogNormal	Ceq Return	Normal
Area	1.0666	1	1.0666	1
Mean	389.3337	389.3337	0.0018	0.0018
Variance	529.3117	529.3117	0.0036	0.0036
Skewness	-0.4346	0.1775	-0.6295	0.0
Kurtosis	3.3084	3.0561	3.5938	3.0

Export Table

Table 2: Table of moments

Chart Tools

In each chart there are interactive tools positioned at the top right. Here are listed all of them starting from the first one.

- Bokeh Logo: hyperlink to access the library used to create all the interactive graphs in the web-application.
- Pan Tool: the pan tool allows the user to pan the plot by left-dragging a mouse or dragging a finger across the plot region.
- Box Zoom: the box zoom tool allows the user to define a rectangular region to zoom the plot bounds too, by left-dragging a mouse, or dragging a finger across the plot area.
- Save: the save tool pops up a modal dialog that allows the user to save a PNG image of the plot.
- Reset: The reset tool turns off all the selected tools.
- Hoover Tool: the hover tool will generate a “tabular” tooltip where each row contains a label, and its associated value.

References

Breeden, D. T., & Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of business*, 621–651.

Shimko, D. (1993). Bounds of probability. *Risk*, *6*(4), 33–37.